Pergamon

Int. J. Solids Structures Vol. 34, No. 14, pp. 1735–1749, 1997

0020-7683/97 $17.00 + .00

PII: S0020-7683(96)00119-9

STABILITY AND BIFURCATION OF FINITE DEFORMATIONS OF ELASTIC CYLINDRICAL MEMBRANES—PART I. STABILITY ANALYSIS

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(*Received* 31 *October* 1995; *in revised form* 10 *June* 1996)

Abstract—The stability of an inflated and extended cylindrical elastic membrane is studied by using an energy stability criterion. Three types of loading devices are considered, that control the internal pressure, the mass of enclosed gas, and the volume of enclosed fluid, respectively. Four boundary conditions are considered for different end clamping conditions. The stability conditions are derived and compared for twelve combinations of these loading devices and boundary conditions. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The stability and bifurcation of inflated cylindrical elastic membranes have been studied by a number of authors using the theory of nonlinear elasticity. A specially interesting feature associated with this problem is the phenomenon of bulging of the membranous tube during the inflation. This phenomenon is often viewed as the onset of instability. Some authors believe that it also represents non-cylindrical solutions bifurcating from the cylindrical solution branch.

Various criteria have been used to derive conditions for stability and bifurcation. A precise and universally accepted criterion for stability does not seem to exist, forcing researchers to employ criteria that are applicable only to special classes of systems. On the other hand, while a precise definition of bifurcation of solution branches is available, a complete bifurcation analysis often presents great mathematical difficulties, and sometimes a less precise definition is used to ease the analysis. To add confusion, stability criteria are sometimes confounded with bifurcation criteria, and vice versa. It is not rare to find in published work that a bifurcation is claimed from onset of instability, and that instability is concluded from the existence of a bifurcation. As an example, the existence of non-trivial solutions to the linearized equilibrium equations has been taken by some as the condition for onset of instability *and* for the existence of bifurcation.

One of the purposes of this two-part paper is to demonstrate the distinctions and connections of instability and bifurcation for the inflation of cylindrical membranes. Although connections do occur between instability and bifurcation, they are after all two totally different phenomena. By using an energy stability criterion and the precise definition of bifurcation, we derive and compare the stability conditions and bifurcation conditions, as well as their relations to the bulging phenomenon. General discussions of the relation between stability and bifurcation can be found in Ogden (1984) and Beatty (1987).

A second purpose of the paper is to study the dependence of the stability and bifurcation conditions on the type of loading devices and boundary conditions. Inflation experiments can be carried out by using a variety of laboratory apparatus, and may display different kinds of instabilities and bifurcations, of which bulging is but one example. In practice, this analysis is useful for designing an inflation experiment when one wishes to either defer or promote instability and bifurcation.

The paper is divided into two parts. Part I is devoted to stability analysis, and part II to bifurcation analysis.

Many have made contributions to the study of the stability of cylindrical deformations of inflated cylindrical membranes. Among others, Cornelinssen and Shield (1961) studied the stability of cylindrical deformations of an elastic membrane with the controlled internal pressure and end displacements. They used a dynamic stability criterion, as well as a static stability criterion. The former criterion is based on the linearization of the equations of motion about the finite cylindrical deformation, and the latter on the linearization of the equations of equilibrium.

Another static stability criterion was employed by Shield (1971, 1972) to study the same problem. This criterion asserts that instability occurs when a certain potential energy, measured from the equilibrium configuration, ceases to be positive definite. Shield found the stable region by determining its boundary where the potential energy becomes semi-definite. This essentially leads to the linearized equations of equilibrium, based on which the earlier static stability criterion used by Cornelinssen and Shield (1961) is defined. Shield considered three different loading devices that control the internal pressure, the mass of an enclosed gas, and the volume of an enclosed incompressible fluid, respectively. He also considered two types of boundary conditions: one with controlled end displacements, and the other with controlled radial end displacements.

To study the progression of the bulge in an elastic tube during inflation, Yin (1977) proposed a model in which the tube with a fully developed bulge consists of cylindrical parts of different radii joined by non-uniformly deformed transition sections. As inflation continues, the shape of each section remains unchanged; a transition section merely moves so that the cylindrical section of larger radius grows in length. Yin showed that under certain constitutive assumptions, this configuration has a lower total energy than a cylindrical configuration, provided the length of the tube is sufficiently large.

Kyriakides and Chang (1991) carried out a detailed experimental and numerical study of the initiation and propagation of a bulge in an elastic tube inflated by pumping in water. Their results agree qualitatively with Yin's model. The fully developed bulge has an approximately cylindrical shape and grows, as inflation continues, in length with little growth in radius.

In Part I of this paper, we use an energy stability criterion to study the stability of cylindrical deformations of an elastic membrane under various loading devices and boundary conditions. The criterion asserts that a deformation is stable if it is a local minimizer of the total energy. This criterion is similar to but weaker than that used by Shield (1971, 1972). The present criterion does not require strict minima at stable deformations, nor that the quadratic energy be positive-definite. In particular, it does not regard as being unstable those deformations that are usually considered to be neutrally stable. In Section 2, we formulate the minimization problem. The equations of equilibrium are derived as the first variation condition. Three types of loading devices and four boundary conditions are introduced. In Section 3, the equations of equilibrium are reduced for cylindrical deformations. The dependences of the state variables on the control variables are examined for various cases. Section 4 is devoted to stability analyses for twelve cases resulted from the combinations of the three loading devices and four boundary conditions. In each case, we derive the stability conditions by solving the integral inequality that is deduced from the second variation condition. The relation between the stability condition and the state variable-control variable dependence is discussed. Finally, a comprehensive comparison is made of the stability conditions for various cases.

2. BASIC EQUATIONS

We consider an elastic membrane that has a circular cylindrical shape with radius R and length L in a reference configuration. Let (R, Θ, Z) and (r, θ, z) be cylindrical material and spatial coordinates, respectively. We shall consider the class of axisymmetric deformations, that can be expressed by

$$r = r(Z), \quad \theta = \Theta, \quad z = z(Z), \quad Z \in [0, L], \tag{1}$$

where $r(Z)$ and $z(Z)$ are smooth functions with $z' > 0$. Here and henceforth, a prime denotes the derivative with respect to Z. The principal stretches of an axisymmetric deformation are given by

$$\lambda_1(Z) = \frac{r(Z)}{R}, \quad \lambda_2(Z) = [r'^2(Z)+z'^2(Z)]^{1/2}. \tag{2}$$

The membrane is assumed to be homogeneous and isotropic in the reference configuration. Its constitutive relation is described by a strain-energy function $W(\lambda_1, \lambda_2)$, whose value corresponds to the strain energy per unit area in the reference configuration. The strain energy stored in the deformed membrane is then given by

$$E_1 = 2\pi R \int_0^L W(\lambda_1, \lambda_2)\,\mathrm{d}Z. \tag{3}$$

The membrane is deformed under the action of a pair of axial forces f applied at the ends and internal pressure supplied by a loading device with a control variable μ. The total potential energy of the loads is

$$E_2 = \phi(V, \mu) - f \int_0^L z'\,\mathrm{d}Z. \tag{4}$$

Here it has been assumed that the potential energy ϕ associated with the internal pressure depends on the parameter μ and the volume V enclosed by the deformed membrane

$$V = \pi \int_0^L r^2 z'\,\mathrm{d}Z. \tag{5}$$

If the loading device controls the internal pressure, the potential function ϕ takes the form

$$\phi(V, \mu) = -V\mu \tag{6}$$

with μ representing the controlled pressure. If, instead, the loading device controls the mass of a gas enclosed in the membrane, the potential function ϕ represents the Helmholtz free energy of the gas, which for an ideal gas is given by

$$\phi(V, \mu) = -k\mu \ln \frac{V}{\mu}, \tag{7}$$

where k is a positive gas constant, and μ now represents the mass of the enclosed gas. In general, we assume, in accordance with basic principles of thermodynamics, that

$$\phi_{VV} \geqslant 0, \quad \phi_{V\mu} < 0, \tag{8}$$

where the subscripts of ϕ denote the partial derivative.

The membrane could also be inflated by an incompressible fluid, its amount being controlled by the loading device. In this case, the potential of internal pressure remains constant for each μ, and we can take, without loss of generality,

$$\phi(V, \mu) = 0. \tag{9}$$

while on the deformed volume V is now imposed a constraint of the form

$$V = \mu, \tag{10}$$

μ this time representing the controlled volume of the fluid.

Four types of boundary conditions will be considered. The first is a free end boundary condition, that imposes no restriction on the deformation at the boundary. The term "free" is used here in kinematical sense, and does not imply that the ends are traction free. The second is an axial displacement boundary condition, in which the axial deformation at the boundary is prescribed by

$$z(0) = 0, \quad z(L) = \bar{z}, \tag{11}$$

and the radial deformation at the boundary is not prescribed. The third is a radial displacement boundary condition, which specifies the radial deformation at the boundary as

$$r(0) = r(L) = \bar{r}, \tag{12}$$

but imposes no restrictions on the axial deformation. The fourth is a fixed end boundary condition which specifies both the axial and radial deformations at the boundary by (11) and (12). These four boundary conditions are schematically illustrated in Fig. 1.

The total energy of the system is given by

$$E = E_1 + E_2. \tag{13}$$

By the energy stability criterion, a stable equilibrium deformation minimizes the total energy in the class of all deformations that satisfy the given boundary condition, and, if applicable, the constraint (10). The first variation condition of this minimization problem

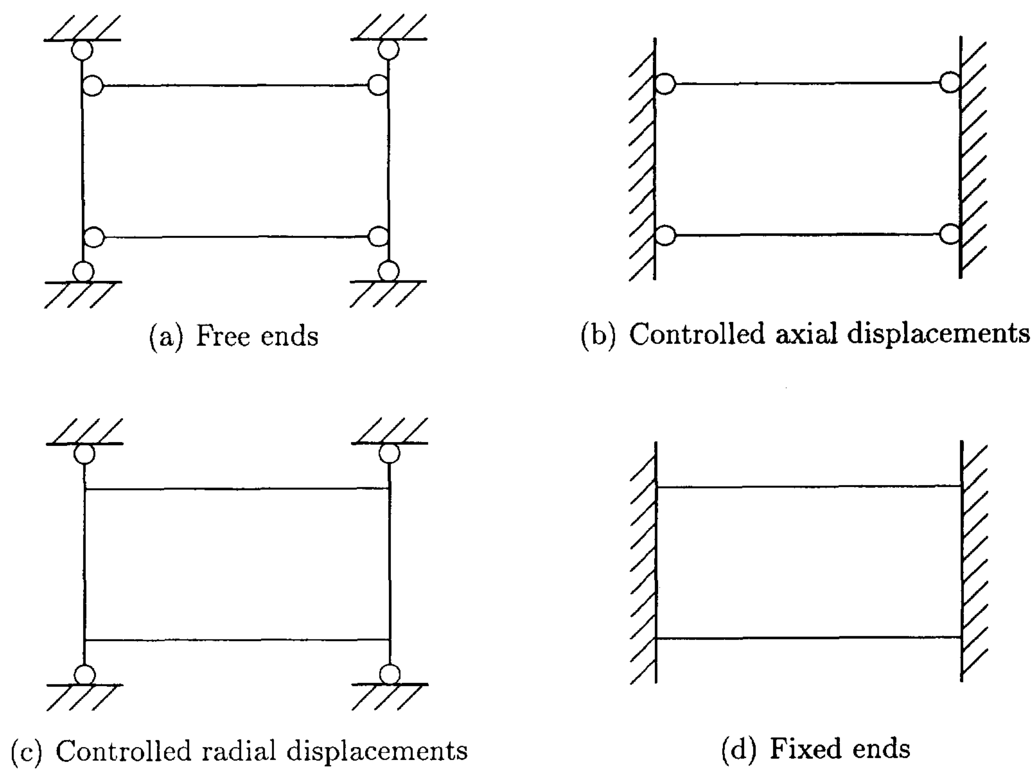

Fig. 1. Four boundary conditions.

leads to the following equations of equilibrium:

$$W_1 - \left(\frac{Rr'W_2}{\lambda_2}\right) - prz' = 0,$$

$$\frac{2\pi Rz'W_2}{\lambda_2} - f - \pi pr^2 = 0, \tag{14}$$

where the subscript i of W denotes the partial derivative with respect to λ_i. The corresponding boundary conditions are

$$r'(0)W_2(\lambda_1(0), \lambda_2(0)) = r'(L)W_2(\lambda_1(L), \lambda_2(L)) = 0 \tag{15}$$

for the free ends, (11) and (15) for the axial displacement controlled ends, (12) for the radial displacement controlled ends, and (11) and (12) for the fixed ends. For the axial displacement controlled ends and the fixed ends, the axial force f cannot be prescribed, and the second term on the right-hand side of (4) can be taken to be zero, with f in $(14)_2$ being a constant of integration which is to be determined by the boundary conditions (11). When the potential function ϕ is of the form (6) or (7), the internal pressure p in (14) is given by

$$p = -\phi_1. \tag{16}$$

On the other hand, when the deformed volume is controlled, the internal pressure p corresponds to a Lagrange multiplier, which is to be determined by the equations of equilibrium (14) and the constraint (10).

3. CYLINDRICAL DEFORMATIONS

A deformation is cylindrical if

$$r(Z) = R\lambda_1, \quad z(Z) = \lambda_2 Z, \tag{17}$$

with constant principal stretches λ_1 and λ_2 along the length. The corresponding deformed volume is

$$V = \pi R^2 L\lambda_1^2\lambda_2. \tag{18}$$

For such a deformation, the equations of equilibrium (14) reduce to algebraic equations

$$W_1 - pR\lambda_1\lambda_2 = 0,$$

$$2\pi RW_2 - f - \pi pR^2\lambda_1^2 = 0. \tag{19}$$

As the control variables μ and f (μ and $\bar{z}$ for the axial displacement controlled ends and the fixed ends) vary in a continuous manner, we assume the existence of a family of cylindrical deformations

$$(\lambda_1 = \lambda_1(\mu, f), \lambda_2 = \lambda_2(\mu, f)), \quad \text{or } (\lambda_1 = \lambda_1(\mu, \bar{z}), \lambda_2 = \lambda_2(\mu, \bar{z})), \tag{20}$$

that depend continuously on the control variables. Here, the same notation has been used for different functions. Specifically, for free ends and radial displacement controlled ends, function $\lambda_1(\mu, f)$ and $\lambda_2(\mu, f)$ in (20) satisfy (18), (19), and either (10) or (16), depending on the type of the loading device. On the other hand, for axial displacement controlled ends and fixed ends, functions $\lambda_1(\mu, \bar{z})$ and $\lambda_2(\mu, \bar{z})$ in (20) satisfy (18), $(19)_1$, either (10) or (16), and the boundary condition

$$\lambda_2(\mu, \bar{z}) = \frac{\bar{z}}{L}. \tag{21}$$

The boundary condition (15) is trivially satisfied by the cylindrical deformation (17). In the cases where the radial displacement boundary condition (12) applies, we shall choose

$$\bar{r} = R\lambda_1(\mu, f) \quad \text{or } \bar{r} = R\lambda_1(\mu, \bar{z}) \tag{22}$$

to satisfy the boundary condition. Such a choice, with the boundary deformation depending on the control variables, appears artificial, and is adopted only to support cylindrical deformations. Nevertheless, we feel that this idealization provides a reasonable approximation to load-independent displacement boundary conditions for large L, and that the corresponding solutions could capture important features of those for more realistic boundary conditions, such as fixing the radial deformation at the value in the reference configuration.

It is interesting to examine how the state variables λ_1, λ_2, V and p depend on the control variables μ and f or $\bar{z}$ for cylindrical deformations, and the connection of such dependences with stability conditions. In the remainder of this section, we derive a few relations of special interest for further reference.

When the axial force and the internal pressure or the mass of the enclosed gas are controlled, the internal pressure p in (19) is given by (16). By substituting (20) into (19) and differentiating with respect to μ and f, we find, formally, the derivatives of λ_1 and λ_2 with respect to μ and f. In particular, we have

$$\frac{\partial \lambda_2}{\partial f} = \frac{\lambda_2^2(a_1 + 4La_5)}{2\pi R[a_1 a_3 - a_2^2 + La_5(a_1 - 4a_2 + 4a_3)]}, \tag{23}$$

where

$$a_1 \equiv \lambda_1^2 W_{11} - \lambda_1 W_1, \quad a_2 \equiv \lambda_1\lambda_2 W_{12} - \lambda_1 W_1, \quad a_3 \equiv \lambda_2^2 W_{22},$$
$$a_4 \equiv \frac{R^2\lambda_1^2 W_2}{\lambda_2}, \quad a_5 \equiv \tfrac{1}{2}\pi R^3 \lambda_1^4 \lambda_2^2 \phi_{VV}. \tag{24}$$

Taking the derivatives of (16) and (18) with respect to μ, and using the previous results, we find that

$$\frac{\partial V}{\partial \mu} = -\frac{\pi R^3 L\lambda_1^4\lambda_2^2(a_1 - 4a_2 + 4a_3)\phi_{V\mu}}{2[a_1 a_3 - a_2^2 + La_5(a_1 - 4a_2 + 4a_3)]}, \quad \frac{\partial p}{\partial \mu} = -\frac{(a_1 a_3 - a_2^2)\phi_{V\mu}}{a_1 a_3 - a_2^2 + La_5(a_1 - 4a_2 + 4a_3)}. \tag{25}$$

When the axial force and the volume of the enclosed fluid are controlled, we can eliminate p in (19) and supplement the resulting equation with (10), V being given by (18). Performing the same analysis as described above, we find that

$$\frac{\partial \lambda_2}{\partial f} = \frac{2\lambda_2^2}{\pi R(a_1 - 4a_2 + 4a_3)}, \quad \frac{\partial p}{\partial \mu} = \frac{2(a_1 a_3 - a_2^2)}{\pi R^3 L\lambda_1^4\lambda_2^2(a_1 - 4a_2 + 4a_3)}. \tag{26}$$

When the axial displacements of ends are controlled, λ_1 and λ_2 are functions of μ and $\bar{z}$. For the pressure or mass control, we differentiate $(19)_1$ and (21) with respect to μ and $\bar{z}$ to find the derivatives of λ_1 and λ_2. In particular,

$$\frac{\partial \lambda_1}{\partial \mu} = -\frac{R\lambda_1^3 \lambda_2 \phi_{V\mu}}{a_1 + 4La_5}. \tag{27}$$

Using this, (18), (16) and $(19)_2$, we also have

$$\frac{\partial V}{\partial \mu} = -\frac{2\pi R^3 L\lambda_1^4 \lambda_2 \phi_{V\mu}}{a_1 + 4La_5}, \quad \frac{\partial p}{\partial \mu} = -\frac{a_1 \phi_{V\mu}}{a_1 + 4La_5},$$

$$\frac{\partial f}{\partial z} = \frac{2\pi R[a_1 a_3 - a_2^2 + La_5(a_1 - 4a_2 + 4a_3)]}{L\lambda_2^2(a_1 + 4La_5)}. \tag{28}$$

Finally, when the axial displacements of ends and the volume are controlled, we differentiate (10) and (21) with respect to μ and z to find the derivatives of λ_1 and λ_2, and further, use (19) to find

$$\frac{\partial p}{\partial \mu} = -\frac{a_1}{2\pi R^3 L\lambda_1^4 \lambda_2^2}, \quad \frac{\partial f}{\partial z} = \frac{\pi R(a_1 - 4a_2 + 4a_3)}{2L\lambda_2^2}. \tag{29}$$

4. STABILITY OF CYLINDRICAL DEFORMATIONS

Before we get into detailed stability analyses, some general observations are in order, in regard to the relations of the stability conditions of various experiments. First, the function (6) defines a sub-potential of the function (7) (or of any potential function that satisfies (8)) in the sense that the variation of the function (7) due to a disturbance of volume from an equilibrium deformation is bounded below by the variation of the function (6) due to the same disturbance. As a result, an equilibrium deformation that minimizes the total energy under pressure control also minimizes the total energy under mass control. That is, a deformation that is stable under pressure control is also stable under mass control. In other words, the experiment of mass control is more stable than that of pressure control.

Further, the volume constraint (10) defines a sub-class of all possible disturbances from an equilibrium deformation. Since the value of the potential function $\phi(V, \alpha)$ does not change in this sub-class of disturbances, the stability condition of an equilibrium deformation under volume control would be the same as that under mass control (or pressure control) with additional kinematic constraint (10). Such a constraint renders a smaller class of admissible deformations and therefore makes the equilibrium deformation more stable. As the result, the experiment of volume control is more stable than those of mass control and pressure control.

By the same argument, the displacement boundary condition (11) or (12) imposes a kinematic constraint upon the class of admissible deformations and consequently tends to stabilize the equilibrium state. Thus, an experiment of controlling either the axial or radial displacements of ends is more stable than that of free ends. Similarly, an experiment of fixing ends with the boundary conditions (11) and (12) is more stable than those of other boundary conditions.

The analyses in the sequel will provide analytic verification of the above observations, as well as additional results, that are not intuitively obvious, concerning the comparison of stabilities of various experiments.

An equilibrium deformation is locally stable if it is a relative minimizer of the total energy in a neighborhood of the deformation. A necessary condition for local stability is that the second variation of the total energy be positive semi-definite. For an equilibrium cylindrical deformation, this condition states that

$$\int_0^L \left[\left(\frac{2W_{11}}{R}-\frac{2W_1}{R\lambda_1}\right)u^2+\left(4W_{12}-\frac{4W_1}{\lambda_2}\right)uv'+2RW_{22}v'^2+\frac{2RW_2}{\lambda_2}u'^2\right]\mathrm{d}Z$$
$$+\pi R^2\lambda_1^2\phi_{VV}\left[\int_0^L(2\lambda_2 u+R\lambda_1 v')\,\mathrm{d}Z\right]^2\geqslant 0 \quad (30)$$

for all smooth functions u and v that satisfy one of the following boundary conditions:

$$W_2u'(0)=W_2u'(L)=0 \quad \text{for free ends;} \quad (31)$$

$$W_2u'(0)=W_2u'(L)=v(0)=v(L)=0 \quad \text{for axial displacement controlled ends;} \quad (32)$$

$$u(0)=u(L)=0 \quad \text{for radial displacement controlled ends;} \quad (33)$$

$$u(0)=u(L)=v(0)=v(L)=0 \quad \text{for fixed ends.} \quad (34)$$

In the case of volume control, u and v also need to satisfy the constraint

$$\int_0^L(2\lambda_2 u+R\lambda_1 v')\,\mathrm{d}Z=0. \quad (35)$$

We note that a strict inequality version of (30) gives a sufficient condition for local stability.

Inequality (30) can be solved by minimizing the left-hand side in u and v', subject to a normalization condition. For convenience, we rewrite inequality (30) as

$$\int_0^L(a_1\hat{u}^2+2a_2\hat{u}\hat{v}'+a_3\hat{v}'^2+a_4\hat{u}'^2)\,\mathrm{d}Z+a_5\left[\int_0^L(2\hat{u}+\hat{v}')\,\mathrm{d}Z\right]^2\geqslant 0, \quad (36)$$

where

$$\hat{u}\equiv\lambda_2 u,\quad \hat{v}\equiv R\lambda_1 v, \quad (37)$$

and the coefficients a_i are given in (24). The boundary conditions (31)–(34) remain unchanged for $\hat{u}$ and $\hat{v}$, while the constraint (35) becomes

$$\int_0^L(2\hat{u}+\hat{v}')\,\mathrm{d}Z=0. \quad (38)$$

An immediate observation is that necessary for inequality (36) to hold are the following conditions

$$a_3\geqslant 0,\quad a_4\geqslant 0. \quad (39)$$

Our task now is to find additional conditions that, along with (39), are equivalent to (36).

Inequality (36) holds if and only if it holds for those $\hat{u}$ and $\hat{v}$ that satisfy

$$\int_0^L(\hat{u}^2+\hat{v}'^2)\,\mathrm{d}Z=1. \quad (40)$$

Under (39) and the normalization condition (40), the minimum of the left-hand side of (36) exists and is attained, when the constraint (38) is not in effect, at $\hat{u}$ and $\hat{v}$ that satisfy

$$(a_1-\alpha)\hat{u}+a_2\hat{v}'-a_4\hat{u}''+2a_5\int_0^L(2\hat{u}+\hat{v}')\,dZ=0,$$

$$a_2\hat{u}+(a_3-\alpha)\hat{v}'+a_5\int_0^L(2\hat{u}+\hat{v}')\,dZ=\beta, \tag{41}$$

where α is a Lagrange multiplier required by the normalization (40), and β a constant of integration resulted from the boundary condition (32) or (34). For the boundary condition (31) or (33), the constant β vanishes.

Since the principal stretches λ_1 and λ_2 are constant for cylindrical deformations, the coefficients of various terms in eqns (41) are constant. Further, the terms involving the integral can be regarded as undetermined constants. Therefore, eqns (41) are a system of linear ordinary differential equations of constant coefficients and can be solved by standard methods. Multiplying $(41)_1$ by $\hat{u}$ and $(41)_2$ by $\hat{v}'$, and integrating the sum, we find that the left-hand side of (36) equals α at a normalized solution of (41). Under the conditions (39), the left-hand side of (36) attains a minimum at a solution of (41). Thus, inequality (36) holds if and only if inequalities (39) hold and $\alpha \geqslant 0$ for all non-trivial solutions (i.e., $\hat{u}$ and $\hat{v}'$ are not both identically zero) of (41) and appropriate boundary conditions. We shall denote by α_{min} the minimum value of α at which eqns (41) have a non-trivial solution. Inequality (36) is then equivalent to (39) and $\alpha_{min} \geqslant 0$. In what follows, we find α_{min} and hence the stability conditions for twelve cases of different loading devices and boundary conditions.

We note that Cases III, IV, VII, VIII, XI and XII below have been studied by Shield (1972), who examined the strict inequality (36). Some of the stability conditions obtained below agree with his. There are, however, some differences because Shield solved eqns (41) with $\alpha = 0$ and assumed that some stability conditions for a plane sheet under dead loading are satisfied, while we derive stability conditions strictly from (36).

Case I. Pressure control and free ends.

In this case, the boundary conditions are (31) with $\beta = 0$ in (41), and the potential function is given by (6) which implies $a_5 = 0$. Equation (38) is not in effect. Under (39), the solution of (41) at $\alpha = \alpha_{min}$ is of the form

$$\hat{u} = C_1, \quad \hat{v}' = C_2. \tag{42}$$

Here and henceforth, C_i are constants. It then follows that α_{min} is the smallest eigenvalue of the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}. \tag{43}$$

Hence, the stability conditions are

$$a_1 \geqslant 0, \quad a_1a_3-a_2^2 \geqslant 0, \tag{44}$$

in addition to (39). By comparing (44) with (23) and (25), keeping in mind that $a_5 = 0$, $\phi_{V\mu} = -1$ in the present case, we find that the cylindrical deformation is stable only if $\partial\lambda_2/\partial f$ and $\partial V/\partial\mu$ are nonnegative. Thus, the deformation is unstable if the deformed length decreases as the axial force increases while holding the pressure constant, or if the deformed volume decreases as the pressure increases while holding the axial force constant.

We note that the form of the solution of (41) may serve to suggest the type of perturbation deformations that induce instability. In the present case, functions $\hat{u}$ and $\hat{v}'$ in (42) indicate a cylindrical perturbation. A bulged deformation is unlikely to occur in this case.

Case II. Pressure control and axial displacement controlled ends.

In this case, we need to solve (41) under the boundary conditions (32). The constant a_5 in (41) is again zero, but β may be non-zero. The solution of (41) at α_{min} is of the form

$$\hat{u} = C_1 \cos\frac{\pi Z}{L} + C_2, \quad \hat{v} = C_3 \sin\frac{\pi Z}{L}, \tag{45}$$

and α_{min} is either a_1 or the smallest eigenvalue of

$$\begin{pmatrix} a_1 + \dfrac{\pi^2 a_4}{L^2} & a_2 \\ a_2 & a_3 \end{pmatrix}. \tag{46}$$

The corresponding stability conditions are (39) and

$$a_1 \geqslant 0, \quad a_1 a_3 - a_2^2 + \frac{\pi^2 a_3 a_4}{L^2} \geqslant 0. \tag{47}$$

It is obvious that under (39) inequalities (44) imply (47). This means, as we have observed, that specifying the axial displacements of the ends tends to stabilize the cylindrical deformation. As is clear from (44), (47) and (24), such a stabilizing effect diminishes as L/R tends to infinity. Also, by comparing (47) with (27) and (28), we find that the cylindrical deformation is stable only if $\partial V/\partial \mu$ is nonnegative, that is, the deformed volume is nondecreasing as the pressure increases while holding the deformed length constant. However, there is no correlation between the stability conditions and the sign of $\partial f/\partial \bar{z}$. This means that the axial force may be decreasing in the deformed length at a stable deformation.

The solution (45) suggests a half-bulge mode of instability. This appears to be consistent with the present boundary condition, that does not restrict radial displacements at the ends.

Case III. Pressure control and radial displacement controlled ends.

The only difference between this case and Case I is that the boundary conditions are now (33). The solution of (41) at α_{min} is of the form

$$\hat{u} = C_1 \sin\frac{\pi Z}{L}, \quad \hat{v}' = C_2 \sin\frac{\pi Z}{L}, \tag{48}$$

and α_{min} is given by the smallest eigenvalue of (46). The corresponding stability conditions are (39) and

$$a_1 + \frac{\pi^2 a_4}{L^2} \geqslant 0, \quad a_1 a_3 - a_2^2 + \frac{\pi^2 a_3 a_4}{L^2} \geqslant 0. \tag{49}$$

Under (39), the conditions (49) are weaker than (47) and (44). As discussed above, it is readily understood that the cylindrical deformation with radial displacement controlled ends should be more stable than that with free ends. However, it does not seem to follow from a simple argument that the deformation with radial displacement controlled ends is more stable than that with axial displacement controlled ends. Also, by comparing (49) with (23) and (25), we find no correlation between the stability conditions and the parameter dependences. In particular, unlike Case I, it is possible to have a stable deformation at which the deformed length would decrease in the axial force, or the deformed volume would decrease in the pressure.

With the radial displacements controlled at ends, it is now no longer possible to have a half bulge as in Case II. Instead, the solution (48) corresponds to a perturbation that is symmetric about the mid-point of the length, that may later develop into a deformation with a bulge at the center.

Case IV. Pressure control and fixed ends.

We now need to solve (41) under the boundary condition (34). Again, we have $a_5 = 0$ in (41), but the constant β can be non-zero. The solution of (41) at α_{min} is of the form

$$\hat{u} = C_1(1-\cos kZ)+C_2 \sin kZ,$$
$$\hat{v} = C_3[L(1-\cos kZ)-(1-\cos kL)Z]+C_4(L\sin kZ - Z\sin kL). \tag{50}$$

For such a solution, α is an eigenvalue of

$$\begin{pmatrix} a_1+a_4k^2 & a_2 \\ a_2 & a_3 \end{pmatrix}, \tag{51}$$

where k satisfies

$$2(1-\cos kL)(a_1+a_4k^2-\alpha)-(a_1-\alpha)kL\sin kL = 0. \tag{52}$$

The last equation along with the condition that the eigenvalues α of (51) be nonnegative defines the stability region in the (a_1, a_2, a_3, a_4) space. The boundary of this region can be determined by solving eqn (52) at $\alpha = 0$, and the equation of vanishing determinant of (51). A detailed analysis shows that the stability conditions in this case are (39) and

$$a_1+k^2a_4 \geqslant 0, \quad a_1a_3-a_2^2+k^2a_3a_4 \geqslant 0, \tag{53}$$

where the parameter k is given by

$$k = \frac{2\pi}{L} \tag{54}$$

when $a_1 \geqslant 0$, and is given by the smallest positive solution of the equation

$$\tan\frac{kL}{2} = \frac{a_1a_3}{a_2^2}\frac{kL}{2} \tag{55}$$

when $a_1 < 0$. It is readily verified that either $k > \pi/L$ or $a_1a_3-a_2^2 > 0$, and hence inequalities (49) imply (53) under (39), confirming the expected result that the cylindrical deformation with fixed ends is more stable than that with radial displacement controlled ends. Also, in the present case, as in Case II, there is no correlation between the stability conditions and the parameter dependences.

It can be readily shown that the solution (50) corresponds to a perturbation that is symmetric about the mid-point and may later develop into a bulged deformation.

Case V. Mass control and free ends.

In this case, we have the potential function (7) (or a function $\phi(V, \mu)$ that satisfies (8)) and the boundary condition (31). The constant β in (41) vanishes. The constraint (38) is again not imposed. The solution of (41) at α_{min} is of the form

$$\hat{u} = C_1\cos\frac{\pi Z}{L}+C_2, \quad \hat{v}' = C_3\cos\frac{\pi Z}{L}+C_4, \tag{56}$$

and α_{min} is the smallest eigenvalue of the matrices (46) and

$$\begin{pmatrix} a_1+4La_5 & a_2+2La_5 \\ a_2+2La_5 & a_3+La_5 \end{pmatrix}. \tag{57}$$

The corresponding stability conditions consist of inequalities (39), (49) and

$$a_1+4La_5 \geqslant 0, \quad a_1a_3-a_2^2+La_5(a_1-4a_2+4a_3) \geqslant 0. \tag{58}$$

By (8), these inequalities imply (39) and (49), and are implied by (39) and (44). Hence, the cylindrical deformation in this case is less stable than that in Case III, but more stable than that in Case I. The latter statement agrees with our earlier observation, while the former does not seem to follow from a simple physical argument. There is not a definite implication relation between (58) and (47). Thus, the cylindrical deformation in the present case may or may not be more stable than that in Case II. Also, by comparing (58) with (23) and (25), keeping (8) in mind, we observe that, similar to Case I, the cylindrical deformation is stable only if the deformed length is non-decreasing in the axial force, and the deformed volume is non-decreasing in the mass of the enclosed gas. However, there is no implication relation between the stability conditions and the sign of $\partial p/\partial\mu$, meaning that it may not be necessary in the present case that the pressure be non-decreasing in the mass at a stable deformation.

The solution (56) again represents a half-bulge perturbation.

Case VI. Mass control and axial displacement controlled ends.

We now need to solve equations (41) subject to the boundary condition (32). The parameter β in (41) may be non-zero. The solution at α_{min} is of the form (45), and α_{min} is either a_1+4La_5 or the smallest eigenvalue of (46). The corresponding stability conditions are (39), (49) and

$$a_1+4La_5 \geqslant 0. \tag{59}$$

This set of inequalities are found to imply (39) and (49), but are implied by (39) and (47), and by (39), (49) and (58). Thus, a cylindrical deformation in this case is more stable, as expected, than that in Cases II and V, but is less stable than that in Case III. This last conclusion is again not physically intuitive. A comparison of (59) with (28), with the aid of (8), shows that at a stable deformation, the deformed volume must be non-decreasing in the mass, but it is not necessary that the pressure be non-decreasing in the mass, nor that the axial force be non-decreasing in the deformed length.

Case VII. Mass control and radial displacement controlled ends.

We now need to solve eqns (41) with $\beta = 0$, subject to the boundary condition (33). The solution at α_{min} is of the form

$$\hat{u} = C_1(1-\cos kZ)+C_2\sin kZ, \quad \hat{v}' = C_3\cos kZ+C_4\sin kZ+C_5. \tag{60}$$

By going through an analysis similar to that in Case IV, we find that α is again given by an eigenvalue of (51), and the stability conditions are (39) and (53), with the parameter k now being given by (54) when

$$a_1a_3-a_2^2+La_5(a_1-4a_2+4a_3) \geqslant 0, \tag{61}$$

and being the smallest positive solution of

$$\tan\frac{kL}{2} = \frac{a_3[a_1a_3-a_2^2+La_5(a_1-4a_2+4a_3)]}{La_5(a_2-2a_3)^2}\frac{kL}{2} \tag{62}$$

when

$$a_1a_3-a_2^2+La_5(a_1-4a_2+4a_3)<0. \tag{63}$$

Since either $k>\pi/L$ or $a_1a_3-a_2^2>0$, the conditions (39) and (53) with k being determined as above are weaker than (39) and (49). Thus, a cylindrical deformation in this case is more stable than it is in case III, as expected. The comparison of the solutions of (62) and (55) is inconclusive, indicating that the cylindrical deformation in the present case may or may not be more stable than that in Case IV. It is also observed that the stability conditions in the present case are not related to the parameter dependences described in (23) and (25).

Once again, the solution (60) represents a symmetric perturbation.

Case VIII. Mass control and fixed ends.

The only difference between this case and Case IV is that a_5 is now non-zero (positive). The analysis in Case IV remains valid for the present case with the addition of a term involving a_5. The solution of (41) at α_{min} is again of the form (50), and the stability conditions are again (39) and (53), with the parameter k being given by (54) if $a_1+4La_5\geqslant 0$, and given by the smallest positive solution of

$$\tan\frac{kL}{2}=\frac{a_1a_3+4La_3a_5}{a_2^2+4La_3a_5}\frac{kL}{2} \tag{64}$$

otherwise. The smallest positive solution of (64) is found to be not smaller than those of (55) and (62) under the condition $a_1a_3-a_2^2\leqslant 0$. As a result, the cylindrical deformation in this case is more stable than those in Cases IV and VII, or in any previous case. Again, the present stability conditions are not related to the signs of the derivatives of state variables in (27) and (28).

Case IX. Volume control and free ends.

In this and the remaining three cases, we have eqn (9) and the constraint (10) in effect. Stability conditions can be derived by solving inequality (36) with $a_5=0$, subject to the constraint (38). This can be done by again minimizing the left-hand side of (36) subject to (38) and the normalization condition (40), yielding

$$(a_1-\alpha)\hat{u}+a_2\hat{v}'-a_2\hat{u}''=2\gamma,\quad a_2\hat{u}+(a_3-\alpha)\hat{v}'=\gamma+\beta, \tag{65}$$

where γ is a Lagrange multiplier required by the constraint (38). In the present case, the boundary condition is (31), and β in (65) vanishes. The solution of (65) at α_{min} is of the form

$$\hat{u}=C_1\cos\frac{\pi Z}{L}+C_2,\quad \hat{v}'=C_3\cos\frac{\pi Z}{L}-2C_2, \tag{66}$$

and α_{min} is given either by $(a_1-4a_2+4a_3)/5$ or by the smallest eigenvalue of (46). The corresponding stability conditions consist of inequalities (39), (49) and

$$a_1-4a_2+4a_3\geqslant 0. \tag{67}$$

Since the conditions (39) and (67) are weaker than (39) and (58), the cylindrical deformation under volume control and the free end boundary condition is more stable, as expected, than that under mass control or pressure control. But, it is less stable than that with radial displacement controlled ends even under pressure control, which is again not physically intuitive. Moreover, it may or may not be more stable than that with axial displacement controlled ends under pressure or mass control. Comparing (67) with (26), we observe that at a stable deformation the deformed length is non-decreasing in the axial force; however, the pressure may or may not be increasing in the controlled volume.

Like in Case V, the solution (66) represents a half-bulge perturbation.

Case X. Volume control and axial displacement controlled ends.

We need to solve eqns (65) and (38) subject to the boundary condition (32). The constant β may be non-zero. The solution at α_{min} is of the form (45), and α_{min} is the smallest eigenvalue of (46). The corresponding stability conditions are (39) and (49). Clearly, these stability conditions are weaker than those in Cases VI and IX, as has been predicted earlier. Moreover, the present stability conditions are found to be identical to those in Case III. In other words, the cylindrical deformation under pressure control with radial displacement controlled ends is equally stable to that under volume control with axial displacement controlled ends. This appears to be a coincidence with no convincing physical explanation. Furthermore, the stability conditions are found not related to the parameter dependences (29). At a stable deformation, the pressure may be decreasing in the volume, and the axial force may be decreasing in the length.

Case XI. Volume control and radial displacement controlled ends.

We need to solve eqns (65) at $\beta = 0$, (38), and the boundary conditions (33). The analysis in this case is parallel to that of Case VII. The form of solution (60), the matrix (51) of which the eigenvalues determine α, and the stability conditions (39) and (53) all remain valid for the present case, with the only difference being that the parameter k in (51) and (53) is now given by (54) when inequality (67) holds, and given by the smallest positive solution of

$$\tan\frac{kL}{2} = \frac{a_3(a_1 - 4a_2 + 4a_3)}{(a_2 - 2a_3)^2}\frac{kL}{2} \tag{68}$$

otherwise. Since the smallest positive solution of (68) is not smaller, under (39), than that of (62), the stability conditions of the present case are weaker than those of Case VII, that is, the cylindrical deformation in this case is more stable than that in Case VII. A comparison of the present stability conditions with those in Case VIII does not lead to a definite conclusion. Further, in the present case, there is no correlation between the stability conditions and the parameter dependences described in (26).

Case XII. Volume control and fixed ends.

By the earlier observations, we expect the cylindrical deformation to be the most stable in this case. Indeed, the solution of (65) at α_{min}, subject to (38) and the boundary condition (34) is of the form

$$\hat{u} = C_1 \sin\frac{2\pi Z}{L}, \quad \hat{v} = C_2\left(1 - \cos\frac{2\pi Z}{L}\right), \tag{69}$$

and α_{min} is the smallest eigenvalue of

$$\begin{pmatrix} a_1 + \dfrac{4\pi^2 a_4}{L^2} & a_2 \\ a_2 & a_3 \end{pmatrix}. \tag{70}$$

The corresponding stability conditions are (39) and

$$a_1 + \frac{4\pi^2 a_4}{L^2} \geqslant 0, \quad a_1 a_3 - a_2^2 + \frac{4\pi^2 a_3 a_4}{L^2} \geqslant 0, \tag{71}$$

which are weaker than the stability conditions of all the previous cases. Again, the present stability conditions are independent of the parameter dependences (29).

It is interesting to note that the solution (69) corresponds to a perturbation with a bulge in one half of the length, and a shrink in the other half. We attribute this seemingly unusual feature to the boundary condition (12) and $(22)_2$, which requires increasing the boundary radius during the inflation so as to maintain the cylindrical deformation.

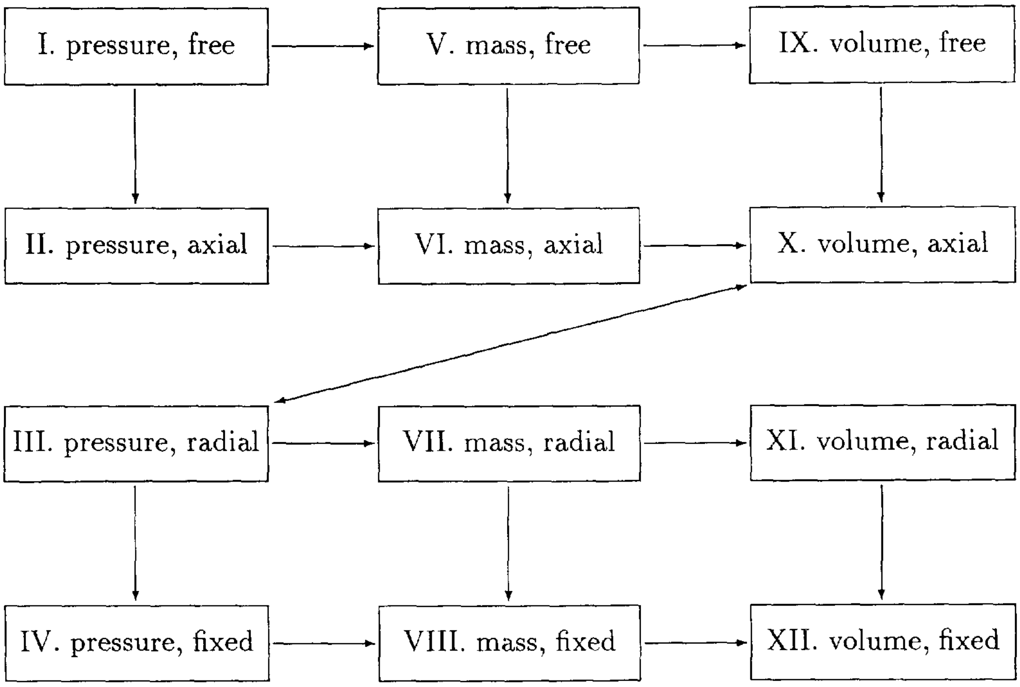

Fig. 2. Comparison of stability conditions.

To summarize the above results, we list in Fig. 2 the comparisons of the stability conditions of all twelve cases. Arrows indicate the implication of stability conditions, and therefore an increasing direction in stability of the cylindrical deformation.

It is observed that the experiment of pressure control and free ends is the least stable. Imposing the radial displacement boundary condition appears to be an effective way to stabilize the cylindrical deformation. Controlling the volume is more effective than controlling the mass in stabilizing the deformation. Imposing the axial displacement boundary condition also stabilizes the deformation, though not as effective as the radial displacement boundary condition does. All these stabilizing effects decrease with the ratio R/L, and all stability conditions converge to (39) and (44) as L/R tends to infinity.

To conclude Part I of this paper, we remark that no definite conclusion can be made on bifurcation from the stability conditions we obtained above. When a cylindrical deformation becomes unstable, a bifurcation may or may not occur. The solution we obtained in solving the eigenvalue problem only represents the perturbation that induces instability. A neighboring non-cylindrical equilibrium deformation may not exist at all. In this case, the membrane will undergo a dynamic process to reach failure or another equilibrium state at distance, as is often believed to occur in Case I. On the other hand, a bifurcation may occur at a neutrally stable deformation. We shall address the bifurcation issue in Part II of this paper.

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